LATTICE VERTEX OPERATOR ALGEBRA $V_{\sqrt{2}\, E_8}$ AND AN ALGEBRA OF MIYAMOTO OF CENTRAL CHARGE $\frac{1}{2}+\frac{21}{22}$ (Algebraic Combinatorics)

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LATTICE VERTEX OPERATOR ALGEBRA $V_{\sqrt{2}E_8}$ AND AN ALGEBRA OF MIYAMOTO OF CENTRAL CHARGE $\frac{1}{2}+\frac{21}{22}$

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ABSTRACT. Motivated by a work of Miyamoto [17], we construct a vertex operator algebra U of central charge $\frac{1}{2}+\frac{21}{22}$ which has the full automorphism group isomorphic to the symmetry group S_3. Actually, we show that the lattice vertex operator algebra $V_{\sqrt{2}E_8}$ contains a subalgebra isomorphic to a tensor product of unitary Virasoro vertex operator algebras $\mathfrak{T}=L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\otimes L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)\otimes L(\frac{25}{28},0)\otimes L(\frac{11}{12},0)\otimes L(\frac{14}{15},0)\otimes L(\frac{52}{55},0)\otimes L(\frac{1}{2},0)\otimes L(\frac{21}{22},0)$ and U is a certain coset subalgebra of $V_{\sqrt{2}E_8}$. We also show that U contains exactly 3 conformal vectors of central charge 1/2 and the inner product between any two of them is $1/2^8$.

1. INTRODUCTION

This work is motivated by a recent article of Miyamoto [17]. In [17], Miyamoto studied a class of vertex operator algebra(VOA) generated by two rational conformal vectors e and f of central charge 1/2. Among other things, he showed that if the inner product $\langle e,f\rangle$ is equal to $\frac{1}{2^8}$, then the vertex operator algebra U generated by e and f is of central charge 16/11 and U contains a subalgebra isomorphic to $L(\frac{1}{2},0)\otimes L(\frac{21}{22},0)$. Moreover, $\dim U_2=3$ and the full automorphism group of U is isomorphic to the symmetry group S_3. In this paper, we shall construct explicitly a VOA

$$\begin{aligned}U\cong\ & L(\frac{1}{2},0)\otimes L(\frac{21}{22},0)\oplus L(\frac{1}{2},0)\otimes L(\frac{21}{22},8)\\ &\oplus L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{7}{2})\oplus L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{45}{2})\\ &\oplus L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{31}{16})\oplus L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{175}{16}),\end{aligned}$$

in the lattice VOA $V_{\sqrt{2}E_8}$ and show that U satisfies all the properties mentioned in [17]. In fact, we shall show that the lattice VOA $V_{\sqrt{2}E_8}$ contains a subalgebra isomorphic to a tensor product of the unitary Virasoro VOAs

$$\begin{aligned}\mathfrak{T}=\ & L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\otimes L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)\otimes L(\frac{25}{28},0)\\ &\otimes L(\frac{11}{12},0)\otimes L(\frac{14}{15},0)\otimes L(\frac{52}{55},0)\otimes L(\frac{21}{22},0)\otimes L(\frac{1}{2},0),\end{aligned}$$

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and obtain a complete decomposition of $V_{\sqrt{2}E_8}$ into a direct sum of irreducible $\mathfrak{T}$-modules. The VOA U is actually a certain commutant (or coset) subalgebra associated with the above decomposition. We also notice that an automorphism of order 3 obtained from the abelian group $\sqrt{2}E_8/\sqrt{2}A_8$ induces a natural $\mathbb{Z}_{\not{p}}$-action on U. This action together with the usual involution θ induced by -1 will form a group S_3 inside the automorphism group of U. In addition, we determine all conformal vectors of central charge $1/2$ inside U and show that the inner of any two of them is $1/2^8$ as mentioned by Miyamoto.

2. Lattice vertex operator algebra $V_{\sqrt{2}E_8}$

2.1. **The lattice $\sqrt{2}E_8$.** Let $\alpha^0 \dots, \alpha^8$ be vectors in $\mathbb{R}^9$ such that $\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}$ for any $i, j = 0, \dots, 8$ and $L = \mathbb{Z}\alpha^0 \oplus \mathbb{Z}\alpha^1 \oplus \cdots \oplus \mathbb{Z}\alpha^8$. Then L is isomorphic to the orthogonal sum of 9 copies of the root lattice A_1. Let $\beta_i = -\alpha_{i-1} + \alpha_i$, $i = 1, \dots, 8$. Then $N = \mathrm{span}_{\mathbb{Z}}\{\beta_1, \dots, \beta_l\}$ is isomorphic to the lattice $\sqrt{2}A_8$. Let

$$\gamma = \frac{1}{3}\left(2\alpha^0 + 2\alpha^1 + 2\alpha^2 - \alpha^3 - \alpha^4 - \alpha^5 - \alpha^6 - \alpha^7 - \alpha^8\right). \tag{2.1}$$

Then γ belongs to the dual lattice $N^* = \{x \in \mathbb{Q} \otimes_{\mathbb{Z}} N | \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in N\}$ of N and the lattice K generated by γ and N is of rank 8. Moreover, we have

Lemma 2.1. $K \cong \sqrt{2}E_8$

Proof. First, we shall note that $\langle \gamma, \gamma \rangle = 4$ and $K = <\gamma, N> = N \cup (\gamma + N) \cup (-\gamma + N)$. Moreover, $K/N \cong \mathbb{Z}_3$ as an abelian group.

Let $\theta_i = \frac{1}{\sqrt{2}}\beta_i = \frac{1}{\sqrt{2}}(-\alpha_{i-1} + \alpha_i)$ for $i = 1, \dots, 7$ and $\theta_8 = \frac{1}{\sqrt{2}}\gamma$. Then

$$\begin{aligned}
&\langle \theta_i, \theta_i \rangle = 2 && \text{for } i = 1, \dots 8,\\
&\langle \theta_{i-1}, \theta_i \rangle = -1 && \text{for } i = 2, \dots 7,\\
&\langle \theta_3, \theta_8 \rangle = -1, && \text{and}\\
&\langle \theta_i, \theta_j \rangle = 0 && \text{for all other } 1 \le i, j \le 8.
\end{aligned}$$

In other words, $\{\theta_1, \dots, \theta_8\}$ is a set of simple roots of the root lattice E_8 and hence

$$K \supset \mathrm{span}_{\mathbb{Z}}\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \gamma\} \cong \sqrt{2}E_8.$$

Since $|K/N| = 3 = |\sqrt{2}E_8/\sqrt{2}A_8|$, $K = \mathrm{span}_{\mathbb{Z}}\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \gamma\} \cong \sqrt{2}E_8$. □

Hence we also know that the vertex operator algebra

$$V_{\sqrt{2}E_8} \cong V_K = V_N \oplus V_{\gamma+N} \oplus V_{-\gamma+N}.$$

2.2. **Conformal vectors in $V_{\sqrt{2}E_8}$.** In this section, we shall study some conformal vectors in $V_{\sqrt{2}E_8}$. We shall show that the Virasoro element of the VOA $V_{\sqrt{2}E_8}$ can be decomposed into a sum of 10 mutually orthogonal conformal vectors $\tilde{\omega}^1, \dots, \tilde{\omega}^{10}$ and the central charge of $c(\tilde{\omega}^i)$ of $\tilde{\omega}^i$ are given by

$$c(\tilde{\omega}^i) = 1 - \frac{6}{(i+2)(i+3)} \qquad \text{for } 1 \le i \le 8,$$
$$c(\tilde{\omega}^9) = \frac{1}{2}, \qquad \text{and} \qquad c(\tilde{\omega}^{10}) = \frac{21}{22}.$$

First, let us recall a construction of certain conformal vectors in $V_{\sqrt{2}A_l}$ from Dong et. al.[4]. Let Φ be the root system of A_l and Φ^+ and Φ^- the set of all positive roots and negative roots, respectively. Then

$$\Phi = \Phi^+ \cup \Phi^- = \Phi^+ \cup (-\Phi^+).$$

Consider a chain of root systems

$$\Phi = \Phi_l \supset \Phi_{l-1} \supset \cdots \supset \Phi_1$$

such that Φ_i is a root system of type A_i. For any $i = 1, 2, \dots, l$, define

$$s^i = \frac{1}{2(i+3)} \sum_{\alpha \in \Phi_i^+} \left(\alpha(-1)^2 \cdot 1 - 2(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha})\right)$$

and

$$\omega = \frac{1}{2(l+1)} \sum_{\alpha \in \Phi_l^+} \alpha(-1)^2 \cdot 1.$$

It was shown by Dong et. al. [4] that the elements

$$\omega^1 = s^1, \qquad \omega^i = s^i - s^{i-1},\ 2 \le i \le l, \qquad \omega^{l+1} = \omega - s^l \tag{2.2}$$

are mutually orthogonal conformal vectors in $V_{\sqrt{2}A_l}$. The subalgebra $\mathrm{Vir}(\omega^i)$ of the vertex operator algebra $V_{\sqrt{2}A_l}$ generated by ω^i is isomorphic to the Virasoro vertex operator algebra $L(c(\omega^i), 0)$ which is the irreducible highest weight module for the Virasoro algebra with central charge $c(\omega^i)$ and highest weight 0 and the central charge $c(\omega^i)$ of ω^i are given by

$$c(\omega^i) = 1 - \frac{6}{(i+2)(i+3)} \text{ for } 1 \le i \le l \quad \text{and} \quad c(\omega^{l+1}) = \frac{2l}{(l+3)}.$$

Since ω^1, ω^2, $\dots$, ω^{l+1} are mutually orthogonal, the subalgebra T of $V_{\sqrt{2}A_l}$ generated by these conformal vectors is a tensor product of $\mathrm{Vir}(\omega^i)$'s, namely,

$$\begin{aligned} T &= \mathrm{Vir}(\omega^1) \otimes \cdots \otimes \mathrm{Vir}(\omega^{l+1}) \\ &\cong L(c(\omega^1), 0) \otimes \cdots \otimes L(c(\omega^{l+1}), 0). \end{aligned}$$

Moreover, $V_{\sqrt{2}A_l}$ is completely reducible as a T-module.

For $l = 8$, there are 9 mutually orthogonal conformal vectors $\omega^1, \dots, \omega^9$ in $V_{\sqrt{2}A_8}$ and the central charge of $\omega^1, \dots, \omega^9$ are $\frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{6}{7}, \frac{25}{28}, \frac{11}{12}, \frac{14}{15}, \frac{52}{55}$ and $\frac{16}{11}$, respectively. In other words, $V_{\sqrt{2}A_8}$ contains a subalgebra isomorphic to

$$T = L(\frac{1}{2},0) \otimes L(\frac{7}{10},0) \otimes L(\frac{4}{5},0) \otimes L(\frac{6}{7},0) \otimes L(\frac{25}{28},0) \\ \otimes L(\frac{11}{12},0) \otimes L(\frac{14}{15},0) \otimes L(\frac{52}{55},0) \otimes L(\frac{16}{11},0)$$

The following lemma can be obtained by direct calculation.

Lemma 2.2. *Let γ be defined as in (2.1) and let*

$$a^1 = \sum_{\substack{\alpha \in (\gamma+\sqrt{2}A_8), \\ \langle \alpha,\alpha \rangle = 4}} e^\alpha \in V_{\gamma+\sqrt{2}A_8} \quad \text{and}$$

$$a^2 = \sum_{\substack{\alpha \in (-\gamma+\sqrt{2}A_8), \\ \langle \alpha,\alpha \rangle = 4}} e^\alpha \in V_{-\gamma+\sqrt{2}A_8}.$$

Then a^1 and a^2 are both highest weight vectors of weight $(0,0,0,0,0,0,0,0,2)$ with respect to the action of T.

Lemma 2.3. *Let $u = a^1 + a^2 = \sum_{\substack{\alpha \in (\gamma+\sqrt{2}A_8), \\ \langle \alpha,\alpha \rangle = 4}} (e^\alpha + e^{-\alpha})$. Then*

$$\tilde{\omega}^9 = \frac{11}{32}\omega^9 + \frac{1}{32}u \quad \text{and} \quad \tilde{\omega}^{10} = \frac{21}{32}\omega^9 - \frac{1}{32}u$$

are mutually orthogonal conformal vectors of central charge $1/2$ and $21/22$, respectively. Moreover, they are orthogonal to $\omega^1, \dots, \omega^8$.

Proof. First, we shall note that for any α, β with square norm 4,

$$(e^\alpha)_1 e^\beta = \begin{cases} e^{\alpha+\beta} & \text{if } \langle \alpha,\beta \rangle = -2 \\ \alpha(-1)^2 & \text{if } \alpha = -\beta \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

and

$$\langle e^\alpha, e^\beta \rangle = (e^\alpha)_3 e^\beta = \begin{cases} 1 & \text{if } \alpha = -\beta \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

Then by direct computation, we have

$$u_1 u = 2(231\omega^9 + 10u), \quad \omega^9_1 \omega^9 = 2\omega^9 \quad \text{and} \quad \omega^9_1 u = 2u.$$

Now, it is easy to verify that both $\tilde{\omega}^9$ and $\tilde{\omega}^{10}$ are conformal vectors.

Since $\sqrt{2}A_8$ has exactly 72 vectors of square norm 4 and $\gamma+\sqrt{2}A_8$ and $-\gamma+\sqrt{2}A_8$ each has 84 vectors of square norm 4, we also have

$$\langle \omega^9, \omega^9\rangle = \frac{8}{11}, \quad \langle \omega^9, u\rangle = 0, \quad \text{and} \quad \langle u, u\rangle = 168. \tag{2.5}$$

Therefore,

$$\langle \tilde{\omega}^9, \tilde{\omega}^9\rangle = \frac{1}{4}, \quad \langle \tilde{\omega}^9, \tilde{\omega}^{10}\rangle = 0, \quad \text{and} \quad \langle \tilde{\omega}^{10}, \tilde{\omega}^{10}\rangle = \frac{21}{44}$$

and hence $\tilde{\omega}^9$ and $\tilde{\omega}^{10}$ are mutually orthogonal conformal vectors of central charge $1/2$ and $21/22$. By the definition, it is also clear that $\tilde{\omega}^9$ and $\tilde{\omega}^{10}$ are orthogonal to $\{\omega^1,\dots,\omega^8\}$ as ω^9 and u are orthogonal to $\{\omega^1,\dots,\omega^8\}$. □

As a corollary, we have

Corollary 2.4. *The lattice VOA $V_{\sqrt{2}E_8}$ contains a subalgebra isomorphic to*

$$\begin{aligned}\mathfrak{T} = {} & L(\tfrac{1}{2},0)\otimes L(\tfrac{7}{10},0)\otimes L(\tfrac{4}{5},0)\otimes L(\tfrac{6}{7},0)\otimes L(\tfrac{25}{28},0)\\ & \otimes L(\tfrac{11}{12},0)\otimes L(\tfrac{14}{15},0)\otimes L(\tfrac{52}{55},0)\otimes L(\tfrac{1}{2},0)\otimes L(\tfrac{21}{22},0),\end{aligned}$$

Proof. Let $\tilde{\omega}^i=\omega^i$ for $i=1,2,\dots,8$. Then $\{\tilde{\omega}^1,\dots,\tilde{\omega}^{10}\}$ is a set of mutually orthogonal conformal vectors of central charge $\frac{1}{2},\frac{7}{10},\frac{4}{5},\frac{6}{7},\frac{25}{28},\frac{11}{12},\frac{14}{15},\frac{52}{55},\frac{1}{2}$ and $\frac{21}{22}$, respectively. Hence, the subalgebra generated by $\{\tilde{\omega}^1,\dots,\tilde{\omega}^{10}\}$ is isomorphic to $\mathfrak{T}$. □

Remark 2.5. Note that the vector $v=a^1-a^2$ is a highest weight vector of weight $(0,0,0,0,0,0,0,1/16,31/16)$ with respect to $\mathfrak{T}$.

2.3. **Decomposition of $V_{\sqrt{2}E_8}$ as $\mathfrak{T}$-submodules.** Next, we shall study the decomposition of $V_{\sqrt{2}E_8}$ as a direct sum of $\mathfrak{T}$-modules. First, let us recall the following theorem from [13].

Theorem 2.6. *The lattice VOA $V_{\sqrt{2}A_8}$ can be decomposed as*

$$V_{\sqrt{2}A_8}\cong V_N\cong \bigoplus_{\substack{0\le k_j\le j+1,\\ j=0,\dots,8,\\ k_j\equiv 0 \mod 2}} L(c_1,h^1_{k_0+1,k_1+1})\otimes\cdots L(c_l,h^8_{k_7+1,k_8})\otimes W(k_8), \tag{2.6}$$

where $W(0)$ is a simple VOA, known as parafermion algebra or W-algebra, of central charge $16/11$ and $W(k), k=0,2,4,6,8$, are irreducible $W(0)$-modules.

Since $V_{\gamma+\sqrt{2}A_8}$ and $V_{-\gamma+\sqrt{2}A_8}$ are irreducible $V_{\sqrt{2}A_8}$-modules and both of them contain highest weight vectors of weight $(0,0,0,0,0,0,0,0,2)$ with respect to T, we also have

$$V_{\gamma+\sqrt{2}A_8} \cong \bigoplus_{\substack{0\le k_j\le j+1,\\ j=0,\dots,8,\\ k_j\equiv 0 \mod 2}} L(c_1, h^1_{k_0+1,k_1+1}) \otimes \cdots L(c_l, h^8_{k_7+1,k_8}) \otimes P(k_8), \tag{2.7}$$

$$V_{-\gamma+\sqrt{2}A_8} \cong \bigoplus_{\substack{0\le k_j\le j+1,\\ j=0,\dots,8,\\ k_j\equiv 0 \mod 2}} L(c_1, h^1_{k_0+1,k_1+1}) \otimes \cdots L(c_l, h^8_{k_7+1,k_8}) \otimes Q(k_8), \tag{2.8}$$

where $P(k_l)$ and $Q(k_l)$ are irreducible $W(0)$-modules whose structure are yet to be determined.

Now, let $U = U(0) = \{V \in V_{\sqrt{2}E_8} \mid (\widetilde{w}^i)_1 v = 0 \ for\ i = 1, 2, \dots, 8\}$. Then, U is a VOA of central charge 16/11 and by combining Corollary 2.4 and (2.6 – 2.8), we have

Theorem 2.7. *The lattice VOA $V_{\sqrt{2}E_8}$ can be decomposed as*

$$V_{\sqrt{2}E_8} \cong \bigoplus_{\substack{0\le k_j\le j+1,\\ j=0,\dots,8,\\ k_j\equiv 0 \mod 2}} L(c_1, h^1_{k_0+1,k_1+1}) \otimes \cdots L(c_l, h^l_{k_7+1,k_8+1}) \otimes U(k_8), \tag{2.9}$$

where $U(k) = W(k) + P(k) + Q(k)$, $k = 0, 2, 4, 6, 8$, are $U(0)$ – modules.

Remark 2.8. Let σ be an automorphism of $V_{\sqrt{2}E_8}$ defined by

$$\sigma(u) = e^{\frac{2\pi i}{3}\langle\gamma,\beta\rangle} \qquad \text{for any} \quad u \in M(1) \otimes e^{\beta} \subset V_{\sqrt{2}E_8}.$$

and let θ be an automorphism of $V_{\sqrt{2}E_8}$ induces by the isometry $\beta \to -\beta$ of $\sqrt{2}E_8$. Then the subgroup generated by σ and θ is isomorphic to S_3. Moreover, σ and θ induce some nontrivial automorphisms of order 3 and order 2 on the subVOA $U(0)$ respectively. In fact, they induce automorphisms of order 3 and order 2 on the submodules $U(k)$, $k = 0, 2, 4, 6, 8$, also. By abuse of notation, we shall still denote them by σ and θ.

Note also that the automorphism σ is in fact induced from the order 3 symmetry among the 3 cosets of $\sqrt{2}A_8$ in $\sqrt{2}E_8$.

Next let us determine the structure of $U(0)$. Since $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$ is rational and contained in $U(0)$, $U(0)$ and $U(k), k = 2, 4, 6, 8$, are direct sum of irreducible $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$-modules. On the other hand,

$$L(\frac{1}{2},0)\otimes L(\frac{21}{22},0), \quad L(\frac{1}{2},0)\otimes L(\frac{21}{22},8), \quad L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{7}{2}),$$
$$L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{45}{2}), \quad L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{31}{16}), \quad \text{and} \quad L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{175}{16}),$$

are the only irreducible modules of $L(\frac{1}{2},0)\otimes L(\frac{21}{22},0)$ which have integral weights.
Hence,

$$\begin{aligned} U(0) = {} & A_1L(\frac{1}{2},0)\otimes L(\frac{21}{22},0)\oplus A_2L(\frac{1}{2},0)\otimes L(\frac{21}{22},8) \\ & \oplus A_3L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{7}{2})\oplus A_4L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{45}{2}) \\ & \oplus A_5L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{31}{16})\oplus A_6L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{175}{16}), \end{aligned}$$

where $A_1,\ldots,A_6$ are the multiplicities of the irreducible summands.
Similarly, we also have

$$\begin{aligned} U(2) = {} & B_1L(\frac{1}{2},0)\otimes L(\frac{21}{22},\frac{13}{11})\oplus B_2L(\frac{1}{2},0)\otimes L(\frac{21}{22},\frac{35}{11}) \\ & \oplus B_3L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{15}{22})\oplus B_4L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{301}{22}) \\ & \oplus B_5L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{21}{176})\oplus B_6L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{901}{176}), \end{aligned}$$

$$\begin{aligned} U(4) = {} & C_1L(\frac{1}{2},0)\otimes L(\frac{21}{22},\frac{50}{11})\oplus C_2L(\frac{1}{2},0)\otimes L(\frac{21}{22},\frac{6}{11}) \\ & \oplus C_3L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{1}{22})\oplus C_4L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{155}{22}) \\ & \oplus C_5L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{85}{176})\oplus C_6L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{261}{176}), \end{aligned}$$

$$\begin{aligned} U(6) = {} & D_1L(\frac{1}{2},0)\otimes L(\frac{21}{22},\frac{111}{11})\oplus D_2L(\frac{1}{2},0)\otimes L(\frac{21}{22},\frac{1}{11}) \\ & \oplus D_3L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{35}{22})\oplus D_4L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{57}{22}) \\ & \oplus D_5L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{533}{176})\oplus D_6L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{5}{176}), \end{aligned}$$

and

$$\begin{aligned} U(8) = {} & E_1L(\frac{1}{2},0)\otimes L(\frac{21}{22},\frac{196}{11})\oplus E_2L(\frac{1}{2},0)\otimes L(\frac{21}{22},\frac{20}{11}) \\ & \oplus E_3L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{117}{22})\oplus E_4L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{7}{22}) \\ & \oplus E_5L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{1365}{176})\oplus E_6L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{133}{176}), \end{aligned}$$

for some suitable B_i, C_i, D_i and E_i. Note that the weights of $U(2), U(4), U(6)$, and $U(8)$ are $2/11+\mathbb{Z}, 6/11+\mathbb{Z}, 1/11+\mathbb{Z}$, and $9/11+\mathbb{Z}$, respectively.

Now by comparing the characters of the left and the right hand sides of (2.9), we find that all A_i's, B_i's, C_i's, D_i's, and E_i's are equal to 1.

Hence we have

$$
\begin{aligned}
U(0) \cong\ & L(\tfrac{1}{2},0) \otimes L(\tfrac{21}{22},0) \oplus L(\tfrac{1}{2},0) \otimes L(\tfrac{21}{22},8) \\
& \oplus L(\tfrac{1}{2},\tfrac{1}{2}) \otimes L(\tfrac{21}{22},\tfrac{7}{2}) \oplus L(\tfrac{1}{2},\tfrac{1}{2}) \otimes L(\tfrac{21}{22},\tfrac{45}{2}) \\
& \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{21}{22},\tfrac{31}{16}) \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{21}{22},\tfrac{175}{16}),
\end{aligned}
$$

$$
\begin{aligned}
U(2) \cong\ & L(\tfrac{1}{2},0) \otimes L(\tfrac{21}{22},\tfrac{13}{11}) \oplus L(\tfrac{1}{2},0) \otimes L(\tfrac{21}{22},\tfrac{35}{11}) \\
& \oplus L(\tfrac{1}{2},\tfrac{1}{2}) \otimes L(\tfrac{21}{22},\tfrac{15}{22}) \oplus L(\tfrac{1}{2},\tfrac{1}{2}) \otimes L(\tfrac{21}{22},\tfrac{301}{22}) \\
& \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{21}{22},\tfrac{21}{176}) \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{21}{22},\tfrac{901}{176}),
\end{aligned}
$$

$$
\begin{aligned}
U(4) \cong\ & L(\tfrac{1}{2},0) \otimes L(\tfrac{21}{22},\tfrac{50}{11}) \oplus L(\tfrac{1}{2},0) \otimes L(\tfrac{21}{22},\tfrac{6}{11}) \\
& \oplus L(\tfrac{1}{2},\tfrac{1}{2}) \otimes L(\tfrac{21}{22},\tfrac{1}{22}) \oplus L(\tfrac{1}{2},\tfrac{1}{2}) \otimes L(\tfrac{21}{22},\tfrac{155}{22}) \\
& \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{21}{22},\tfrac{85}{176}) \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{21}{22},\tfrac{261}{176}),
\end{aligned}
$$

$$
\begin{aligned}
U(6) \cong\ & L(\tfrac{1}{2},0) \otimes L(\tfrac{21}{22},\tfrac{111}{11}) \oplus L(\tfrac{1}{2},0) \otimes L(\tfrac{21}{22},\tfrac{1}{11}) \\
& \oplus L(\tfrac{1}{2},\tfrac{1}{2}) \otimes L(\tfrac{21}{22},\tfrac{35}{22}) \oplus L(\tfrac{1}{2},\tfrac{1}{2}) \otimes L(\tfrac{21}{22},\tfrac{57}{22}) \\
& \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{21}{22},\tfrac{533}{176}) \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{21}{22},\tfrac{5}{176}),
\end{aligned}
$$

and

$$
\begin{aligned}
U(8) \cong\ & L(\tfrac{1}{2},0) \otimes L(\tfrac{21}{22},\tfrac{196}{11}) \oplus L(\tfrac{1}{2},0) \otimes L(\tfrac{21}{22},\tfrac{20}{11}) \\
& \oplus L(\tfrac{1}{2},\tfrac{1}{2}) \otimes L(\tfrac{21}{22},\tfrac{117}{22}) \oplus L(\tfrac{1}{2},\tfrac{1}{2}) \otimes L(\tfrac{21}{22},\tfrac{7}{22}) \\
& \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{21}{22},\tfrac{1365}{176}) \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{21}{22},\tfrac{133}{176}),
\end{aligned}
$$

Theorem 2.9. *U is a simple VOA and $U(k)$ for $k = 0, 2, 4, 6, 8$ are irreducible U-modules.*

Proof. Since

$$\begin{aligned} U(0) = \; & L(\frac{1}{2},0)\otimes L(\frac{21}{22},0) + L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{21}{22},\frac{45}{2}) \\ & + L(\frac{1}{2},0)\otimes L(\frac{21}{22},8) + L(\frac{1}{2},\frac{1}{2})\otimes L(\frac{1}{2},\frac{7}{2}) \\ & + L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{31}{16}) + L(\frac{1}{2},\frac{1}{16})\otimes L(\frac{21}{22},\frac{175}{16}) \end{aligned}$$

as an $L(\frac{1}{2},0)\otimes L(\frac{21}{22},0)$-module, by the fusion rules, U is clearly simple.

Now, by the fusion rules and the decomposition, it is also clear that $U(k)$ for $k = 0,2,4,6,8$, are irreducible as U-modules. □

3. Conformal vectors in U

In this section, we shall compute all the conformal vectors in U. First, we shall note that $\dim U_2 = 3$ and $\{\tilde{\omega} = \omega^9, u, v\}$ forms a basis of U_2.

Theorem 3.1. *There are exactly 7 conformal vectors in U, namely, the Virasoro element $\tilde{\omega}$ of U, 3 conformal vectors of central charge $1/2$ and 3 conformal vectors of central charge $21/22$.*

Proof. First we shall note that U_2 is spanned by $\{\tilde{\omega}, u, v\}$. Let $x = a\tilde{\omega} + bu + cv$ be a conformal vector in U_2. Then $x_1x = 2x$. Since $\tilde{\omega}_1\tilde{\omega} = 2\tilde{\omega}$, $\tilde{\omega}_1u = 2u$, $\tilde{\omega}_1v = 2v$, $u_1u = 2(231\tilde{\omega} + 10u)$, $u_1v = -20v$, and $v_1v = 2(-231\tilde{\omega} + 10u)$, by direct computation, we know that

$$\begin{aligned} & a^2 + 231b^2 - 231c^2 = a, \\ & 2ab + 10b^2 + 10c^2 = b, \quad \text{and} \\ & 2ac - 20bc = c. \end{aligned} \tag{3.1}$$

Solving the above equations, we obtain 7 non-trivial solutions, namely,

$$\begin{aligned} & \{a = 1, b = 0, c = 0\}, \\ & \{a = \frac{11}{32}, b = \frac{1}{32}, c = 0\}, && \{a = \frac{21}{32}, b = \frac{-1}{32}, c = 0\}, \\ & \{a = \frac{11}{32}, b = \frac{-1}{64}, c = \frac{\sqrt{-3}}{64}\}, && \{a = \frac{21}{32}, b = \frac{1}{64}, c = \frac{\sqrt{-3}}{64}\}, \\ & \{a = \frac{11}{32}, b = \frac{-1}{64}, c = \frac{-\sqrt{-3}}{64}\}, && \{a = \frac{21}{32}, b = \frac{1}{64}, c = \frac{-\sqrt{-3}}{64}\}. \end{aligned}$$

When $\{a = 1, b = 0, c = 0\}$, $x = \tilde{\omega}$ is the Virasoro element of U.

When $\{a = \frac{11}{32}, b = \frac{1}{32}, c = 0\}$, $\{a = \frac{11}{32}, b = \frac{-1}{64}, c = \frac{\sqrt{-3}}{64}\}$, or $\{a = \frac{11}{32}, b = \frac{-1}{64}, c = \frac{-\sqrt{-3}}{64}\}$, $\langle x, x\rangle = 1/4$ and x is a conformal vector of central charge $1/2$.

When $\{a=\frac{21}{32}, b=\frac{-1}{32}, c=0\}$, $\{a=\frac{21}{32}, b=\frac{1}{64}, c=\frac{\sqrt{-3}}{64}\}$, or $\{a=\frac{21}{32}, b=\frac{1}{64}, c=\frac{-\sqrt{-3}}{64}\}$, $\langle x,x\rangle = 21/44$ and x is a conformal vector of central charge $21/22$. □

Lemma 3.2. *Let* $e^1=\frac{11}{32}w^9+\frac{1}{32}u$, $e^2=\frac{11}{32}w^9-\frac{1}{64}u+\frac{\sqrt{-3}}{64}v$, *and* $e^3=\frac{11}{32}w^9-\frac{1}{64}u-\frac{\sqrt{-3}}{64}v$ *be the three rational conformal vectors of central charge* $\frac{1}{2}$ *in* U. *Then* $\langle e^i,e^j\rangle=\frac{1}{2^8}$ *if* $i\neq j$.

Proof. By (2.4), it is easy to show that

$$\langle\omega^9,\omega^9\rangle=\frac{8}{11},\quad \langle u,u\rangle=168,\quad \langle v,v\rangle=-168,$$

and

$$\langle\omega^9,u\rangle=\langle\omega^9,v\rangle=\langle u,v\rangle=0.$$

Thus, we have

$$\langle e^i,e^j\rangle=\begin{cases}1/2^8 & \text{if } i\neq j,\\ 1/4 & \text{if } i=j,\end{cases}$$

as desired. □

Theorem 3.3. *Let* U_2 *be the Griess algebra of* U. *Then* $\operatorname{Aut}U_2\cong S_3$.

Proof. Let g be an element of $\operatorname{Aut}U_2$. Then it will induce a permutation on the three conformal vectors e^1,e^2 and e^3. Since U_2 is generated by e^1,e^2 and e^3, $\operatorname{Aut}U_2$ must itself a permutation subgroup on $\{e^1,e^2,e^3\}$. On the other hand, by our construction, $\operatorname{Aut}U_2$ already contains elements of order 3 and order 2, namely σ and θ. Thus $\operatorname{Aut}U_2\cong S_3$. □

Theorem 3.4. *The full automorphism group of* U *is isomorphic to* S_3.

Proof. Let $g\in\operatorname{Aut}U$ and let G be the subgroup of $\operatorname{Aut}U$ generated by σ and θ. Since

$$\operatorname{Aut}U_2=\{h|_{U_2}\mid h\in G\},$$

there exists an $h\in G$ such that $gh^{-1}|_{U_2}=id_{U_2}$. In particular, $\rho=gh^{-1}$ will fix the conformal vectors $\tilde{\omega}^9$, $\tilde{\omega}^{10}$ and thus fixes the subVOA $L(1/2,0)\otimes L(21/22,0)$. Hence ρ will map highest weight vectors to highest weight vectors of the same type. Moreover in U highest weight vectors are unique (up to scalar multiple) and ρ preserves their inner product. Hence ρ must fix U. Thus $g=h\in G$ and $\operatorname{Aut}U=G\cong S_3$. □

Remark 3.5. Recall from Miyamoto [14] that for each conformal vector e of central charge $1/2$, one can define an automorphism τ_e by

$$\tau_e=\begin{cases}1 & \text{on the summands isomorphic to } L(1/2,0) \text{ or } L(1/2,1/2),\\ -1 & \text{on the summands isomorphic to } L(1/2,1/16).\end{cases}$$

In the VOA U, τ_{e^1} actually corresponds the permutation $e^2 \leftrightarrow e^3$ and τ_{e^2} corresponds to $e^1 \leftrightarrow e^3$. On the other hand, the order 3 automorphism σ corresponds to the cyclic permutation $e^1 \to e^2 \to e^3 \to e^1$. Hence we have

$$\sigma = \tau_{e^2}\tau_{e^1}.$$

References

1. C. Dong, Vertex algebras associated with even lattices, *J. Algebra* **161** (1993), 245-265.
2. C. Dong, C. Lam and H. Yamada, Decomposition of the vertex operator algebra $V_{\sqrt{2}A_3}$, *J. Algebra* **222** (1999), 500-510.
3. C. Dong and J. Lepowsky, *Generalized vertex algebras and relative vertex operators*, Progress in Math. Vol. 112, Birkhäuser, Boston 1993.
4. C. Dong, H. Li, G. Mason and S.P. Norton, Associative subalgebras of Griess algebra and related topics, *Proc. of the Conference on the Monster and Lie algebra at the Ohio State University*, May 1996, ed. by J. Ferrar and K. Harada, Walter de Gruyter, Berlin - New York, 1998.
5. C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, *Pro. Symp. Pure. Math.*, American Math. Soc. 56 II (1994), 295-316.
6. I. B. Frenkel, Y. Huang and J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, Mem. Amer. Math. Soc. 104, 1993.
7. I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Math., Vol. 134, Academic Press, 1988.
8. R. Griess, The Friendly Giant, *Invent. Math.* **69** (1982), 1-102.
9. K. Kitazume, C. Lam and H. Yamada, Decomposition of the Moonshine vertex operator algebra as Virasoro modules, *J. Algebra*, **226** (2000), 893-919.
10. K. Kitazume, C. Lam, and H. Yamada, A class of vertex operator algebras constructed from $\mathbb{Z}_8$ codes, *J. Algebra*, **242** (2001), 338-359.
11. M. Kitazume, M. Miyamoto and H. Yamada, Ternary codes and vertex operator algebras, in *J. Algebra*, **223** (2000), 379-395.
12. C. Lam and H. Yamada, $\mathbb{Z}_2 \times \mathbb{Z}_2$ codes and vertex operator algebras, *J. Algebra* **224** (2000), 268-291.
13. C. Lam and H. Yamada, Decomposition of the lattice vertex operator algebra $V_{\sqrt{2}A_l}$, preprint.
14. M. Miyamoto, Griess algebras and conformal vectors in vertex operator algebras, *J. Algebra* **179** (1996), 523-548.
15. M. Miyamoto, 3-state Potts model and automorphism of vertex operator algebra of order 3, *J. Algebra* **239** (2001), 56–76.
16. M. Miyamoto, A new construction of the moonshine vertex operator algebra over the real number field, preprint.
17. M. Miyamoto, VOAs generated by two conformal vectors whose τ-involutions generate S_3, to appear in J. Algebra.
18. W. Wang, Rationality of Virasoro vertex operator algebras, *Duke Math. J. IMRN*, **Vol. 71**, No. 1 (1993), 197–211.

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